Towards optimal coordination of vehicles at road intersections

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***Abstract*—This paper considers the problem of coordinating a number of vehicles in order to guarantee safe passage through a road intersection. Formulating such a coordination problem in a constrained optimal control framework leads to an optimization problem with a complexity that scales badly with the number of vehicles. Hence, in this paper we consider an optimal scheduling formulation and study the cost and constraints functions of the resulting optimization problem. Interesting properties of the cost and constraints functions are shown that can be used to formulate lower complexity problems leading to suboptimal approximated solution of the original problem.**

I. Introduction

In this extended abstract we consider the coordination of cooperative, autonomous vehicles at a road intersection. In general, aspects of the problem relates to the study of communication networks, environment perception/estimation and control. However, we will focus our attention solely on the latter and in particular on an optimal control formulation of the problem. The coordination scenario we consider is sketched in Figure 1a and can be stated as follows: A set of N vehicles have to cross a road intersection, each traveling along a predefined path. The speed of each vehicle has to be adapted in order to minimize a global cost function (e.g., the sum of local costs) while fulfilling physical constraints and avoiding collisions between vehicles. We start by modeling the road intersection and formulating a constrained optimal control problem, which solves the coordination problem, in Sections II and III, respectively. Even in case of simple vehicle modeling, solving the resulting non-convex constrained optimization problem can be computationally demanding. The collision avoidance constraints introduce a total number of $N!$ possible crossing orders. Hence, the worst case time complexity of any algorithm evaluating all possible crossing orders would grow as $\mathcal{O}(N!)$. Lower complexity approaches have been proposed in [3], where an approximated *feasible* solution is found but optimality is not addressed, or in [2] where the optimality of a generic sequential coordination policy is studied. Similar works are found in [1] or [4]. Our objective in this manuscript is to translate the optimal control problem into an optimal scheduling problem and to study the resulting optimization problem. In Section IV, the coordination problem is broken down into a scheduling (coordination) problem, determining the time instants when each vehicle has to enter and clear the intersection, and N local problems where each vehicle adapts its velocity profile to occupy the intersection in the assigned time slot. The scheduling problem is still a non-convex problem, while the local problems, depending on the local cost functions and constraints, can be cast as standard quadratic programs. After showing equivalence of the optimal scheduling formulation and the original constrained optimal control problem, we focus on the local costs, showing important quasi-convexity properties. In Section V, a numerical example illustrates the obtained results while Section VI closes the paper by explaining how the obtained results can be used to build approximated solutions of the original constrained optimal control problem.

II. Model

We consider a scenario where $N \in \mathbb{Z}_+$ vehicles, termed systems in the remainder of the paper, approach a traffic intersection. The paths of the vehicles through the intersection are predefined and assumed known, due to which we consider only the one dimensional motion along the path.

A. Dynamics

We restrict our study to the LTI case, where the motion of each system $i \in \mathcal{N} = 1, ..., N$ is described on the form

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t), \\ y_i(t) &= C_i x_i(t). \end{aligned} \tag{1}$$

Here, $A_i \in \mathbb{R}^{n_i \times n_i}, B_i \in \mathbb{R}^{n_i \times m_i}$ and $C_i \in \mathbb{R}^{1 \times n_i}$ for some integers $n_i, m_i > 0$, and the scalar output $y_i(t)$ is the position along the path of system i. Further, we assume that $u_i(t)$ is continuous, that the pair (A_i, B_i) is controllable, and that the state and control trajectories are constrained by linear inequalities arising from, e.g., actuator limitations and traffic rules, so that

$$\begin{aligned} x_i(t) \in X_i &= \{y(t) \,| G_i y(t) \leq b_i, \; \forall t\}, \\ u_i(t) \in U_i &= \{v(t) \mid F_i v(t) \leq d_i, \; \forall t\}, \end{aligned} \tag{2}$$

for some $G_i \in \mathbb{R}^{k_i \times n_i}, b_i \in \mathbb{R}^{k_i}, F_i \in \mathbb{R}^{p \times m_i}, d_i \in \mathbb{R}^{p_i}$ and integers $k_i, p_i > 0$. Additionally, we only consider *strongly output monotone* systems, i.e., systems such that

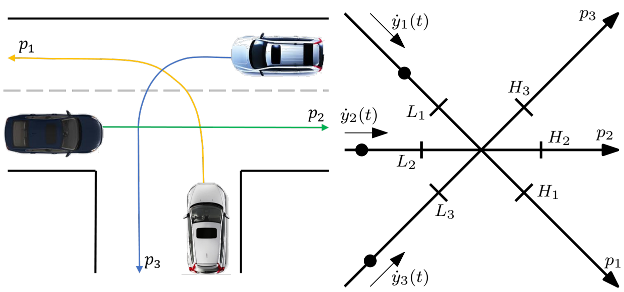
$$\dot{y}_i(t) = C_i \dot{x}_i(t) \geq \varepsilon, \forall t, \tag{3}$$

for some $\varepsilon > 0$. Consequently, given an initial condition $x(0) = x_0$, solutions to (1), for which the above mentioned restrictions are satisfied, belong to the following set of functions

$$D(x_0) = \left\{ \left[x^T(t), u^T(t)\right]^T \;\mid (1), (2), (3), x(0) = x_0 \right\} \tag{4}$$

B. Intersection and Collision Modelling

We model the intersection as a closed and compact subset of positions along the path of each system, defined by a lower limit L_i and a upper limit H_i, as depicted in Figure 1b. A system is



(a) A road intersection scenario with predefined paths p_1, p_2, p_3.

(b) The abstraction of the intersection used in modelling

Fig. 1: A vehicle coordination problem at a cross intersection.

therefore inside the intersection at all times t such that

$$x_i(t) \in \mathcal{E}_i = \{z \mid L_i \leq C_i z \leq H_i\}. \tag{5}$$

It follows from (3) that the time instance at which the system with the state trajectory $x_i(t)$ *enters* the intersection, uniquely can be defined as

$$\tau_i = t : C_i x_i(t) = L_i, \tag{6}$$

and the time instant at which it *exits* the intersection as

$$\xi_i = t : C_i x_i(t) = H_i. \tag{7}$$

For any feasible state trajectory $x_i(t)$ the times t for which (5) holds lie in the closed and compact *occupancy time* interval $[\tau_i, \xi_i]$, i.e., $t \in [\tau_i, \xi_i] \Leftrightarrow x_i(t) \in \mathcal{E}_i$. The requirement for collision avoidance is that two vehicles can not occupy the intersection simultaneously, i.e., that all trajectories $x_i(t), i \in \mathcal{N}$ are such that

$$[\tau_i, \xi_i] \cap [\tau_j, \xi_j] = \emptyset, \ \forall i, j \in \mathcal{N}, i \neq j, \tag{8}$$

or equivalently

$$\left[x_i^T(t), x_j^T(t)\right]^T \notin \mathcal{E}_i \times \mathcal{E}_j, \ \forall t, \ \forall i, j \in \mathcal{N}, i \neq j, \tag{9}$$

where $\times$ denotes the cartesian product.

III. Optimal control formulation

Consider the following *local* performance criteria:

$$J_i(x_i(t), u_i(t)) = \int_0^{t_f} \Lambda_i(x_i(t), u_i(t)) dt, \tag{10}$$

where $\Lambda_i(x_i(t), u_i(t))$ is convex in $x_i(t), u_i(t)$, with fixed time horizon $t_f > 0$, for all $i \in \mathcal{N}$. The problem of finding the optimal state trajectory and control can then be formalized as

$$\min_{x_i(t), u_i(t), i \in \mathcal{N}} \quad \sum_i^N J_i(x_i(t), u_i(t)) \tag{11a}$$

$$\text{s.t.} \quad [x_i^T(t), u_i^T(t)] \in D_i(x_{0,i}), \ \forall i \in \mathcal{N} \tag{11b}$$

$$\mathbf{x}(t) \notin \prod_{i=1}^N \mathcal{E}_i, \ \forall t \tag{11c}$$

where $\mathbf{x}(t) = [x_1^T(t), ..., x_N^T(t)]^T$ and $\prod$ is the cartesian product. Note that the collision avoidance condition (11c) renders the problem non-convex. More precisely, certain regions in the interior of the joint state space $\mathbf{x}(t)$ are excluded from its domain.

IV. Problem Analysis

Consider the following $N+1$ problems:

$$\begin{aligned} \min_{\tau_i, \xi_i, i \in \mathcal{N}} \quad & \sum_i^N F_i(\tau_i, \xi_i) \\ \text{s.t.} \quad & [\tau_i, \xi_i] \cap [\tau_j, \xi_j] = \emptyset, \ \forall i, j \in \mathcal{N}, i \neq j. \end{aligned} \tag{12a}$$

$$\begin{aligned} F_i(\tau_i, \xi_i) = \min_{x_i(t), u_i(t)} \quad & J_i(x_i(t), u_i(t)) \\ \text{s.t.} \quad & [x_i^T(t), u_i^T(t)] \in D_i(x_{0,i}), \\ & C_i x_i(\tau_i) = L_i, \\ & C_i x_i(\xi_i) = H_i \end{aligned} \tag{12b}$$

The *local* problem (12b) establishes how each vehicle should pass the intersection given that a certain occupancy interval $[\tau_i, \xi_i]$ is allocated to it, whereas (12a) is the global *coordination* problem allocating the occupancy time slots to each vehicle. We want to emphasize that $F_i(\tau_i, \xi_i)$ is the optimal value of the objective function in (12b), with minimizers satisfying the entry and exit time requirements. Further, $F_i(\tau_i, \xi_i)$ is defined only for τ_i and ξ_i such that $x(t), u(t)$ feasible in (12b) exist. This means that the domain D_{F_i} of $F_i(\tau_i, \xi_i)$ is implicitly given by the feasible set in the right hand side of (12b).

Lemma 1. *The optimization problems* (11) *and* (12) *are equivalent.*

Proof: (Sketch) First we note that both problems are non-convex and have more than one solution in general. Let $\tau_i^*, \xi_i^*, i \in \mathcal{N}$ be minimizers of (12) and $x_i^*(t), u_i^*(t), i \in \mathcal{N}$ be minimizers of (11). Further, let $\bar{\tau}_i, \bar{\xi}_i$ be the entry and exit times associated with $x_i^*(t), u_i^*(t)$ and $z_i(t), v_i(t)$ be the minimizers in the right hand side of (12b) given τ_i^*, ξ_i^*. Then, as solutions of (12) satisfies (8) and solutions to (11) satisfies (9), and since the two collision avoidance conditions are equivalent, we must have that $\bar{\tau}_i, \bar{\xi}_i$ is feasible in (12) and $z_i(t), v_i(t)$ is feasible in (11). Thus, we can conclude that

$$F_i(\tau_i^*, \xi_i^*) \leq F_i(\bar{\tau}_i, \bar{\xi}_i) \tag{13}$$

$$J_i(x_i(t)^*, u_i^*(t)) \leq J_i(z_i(t), v_i(t)) \tag{14}$$

Then, since by definition $J_i(x_i(t)^*, u_i^*(t)) = F_i(\bar{\tau}_i, \bar{\xi}_i)$ and $J_i(z_i(t), v_i(t)) = F_i(\tau_i^*, \xi_i^*)$, we must have $F_i(\tau_i^*, \xi_i^*) = F_i(\bar{\tau}_i, \bar{\xi}_i)$. Since the relationships between the decision variables in the respective problem are unique, we therefore have that the two problems are equivalent. ■

This implies that given explicit knowledge of $F_i(\tau_i, \xi_i)$ and its domain D_{F_i}, the coordination can be made without direct consideration of the dynamics. To solve (12a) it is therefore paramount to fully understand both $F_i(\tau_i, \xi_i)$ and D_{F_i}. In the following two subsections some of the important properties of D_{F_i} and $F_i(\tau_i, \xi_i)$ will be investigated. For brevity, the system index i will be dropped in the treatment.

A. Domain of $F(\tau, \xi)$

From (2) we have that actuation is limited, whereby we with the monotonicity in (3) can conclude that there must exist an *earliest* and *latest* time at which the intersection can be entered. From this, we can state that $\tau \in [T^l, T^h]$, for some T^l and T^h resulting from application of the maximum and minimum admissible controls respectively. Similarly, for a given $\tau \in [T^l, T^h]$ some degree of freedom exits as to when the intersection is exited. However, the range of possible exit times will clearly change for different τ, due to which we state that $\xi(\tau) \in [g^l(\tau), g^h(\tau)]$, with $g^l(\tau), g^h(\tau)$ resulting from application of maximum and minimum admissible controls on $t \geq \tau$. We can subsequently state that D_F must be such that

$$D_F \subseteq S = \left\{\tau, \xi \,|\tau \in \left[T^l, T^h\right], \xi \in \left[g^l(\tau), g^h(\tau)\right]\right\}, \quad (15)$$

after which we give the following lemma

Lemma 2. $\tau, \xi \in D_F \Leftrightarrow \exists [x(t), u(t)] \in D(x_0)$

The proof is omitted, but follows directly from the convexity of $D(x_0)$ and the monotonicity of $Cx(t)$. The important implication of Lemma 2 is that $D_F = S$ and as such, that it is closed and compact.

B. Characterization of $F(\tau, \xi)$

Considering the function $F(\tau, \xi)$, we first note that it is a composition of continuous functions and hence continuous. Secondly, we present the following, expected result.

Lemma 3. *$F(\tau, \xi)$ has a unique minimum in $[T^l, T^h]$*

The proof is omitted here, but follows directly from the strict convexity of the right hand side in (12b). In preparation of the main result of this section stated in Theorem 1, we state the following reduction of the local problem (12b):

$$\begin{aligned} \min_{x(t),u(t)} \quad & J(x(t), u(t)) \\ \text{s.t.} \quad & [x^T(t), u^T(t)] \in D(x_0) \\ & Cx(\tau) = L. \end{aligned} \quad (16)$$

In (16), only a specified entry time is enforced, and the exit time is unspecified, as opposed to (12b) where both entry and exit times are enforced. Denoting the minimizer of (16) as $x^*_\tau(t), u^*_\tau(t)$, where the subscript denotes the enforced entrance time, we then proceed to define the entry-exit time relationship of the minimizer as a function of τ.

Definition 1. *Optimal exit time $g^*(\tau) = t : Cx^*_\tau(t) = H$.*

It can be shown that $g^*(\tau)$ is continuous and, with τ^*, ξ^* as the unique minimizers of $F(\tau, \xi)$, that $g^*(\tau^*) = \xi^*$. This means that $g^*(\tau)$ passes through the unique minimum of $F(\tau, \xi)$ and from Lemma 3 it then follows that $F(\tau, g^*(\tau))$ has a unique minimum in τ^*. With this we are ready to state the main result of this section:

Theorem 1. *For strongly output monotone systems $F(\tau, g^*(\tau))$, given by* (12b)*, is quasi-convex.*

Proof: (Sketch) Firstly, we will show that $F(\tau, g^*(\tau))$ is monotonically increasing in deviations from its minimum in τ^*. Let $\tau^* < \alpha < \beta < T^H$, and define a relaxation of (16) as

$$\begin{aligned} \underset{x(t),u(t)}{\text{minimize}} \quad & J(x(t), u(t)) \\ \text{s.t.} \quad & [x^T(t), u^T(t)] \in D(x_0), \\ & Cx(\tau) \leq L, \end{aligned} \quad (17)$$

i.e., a problem which allows *later* entry times, and denote its optimal trajectories for the state and control $z^*_\tau(t)$ and $v^*_\tau(t)$ respectively. Note the the subscript indicates the entrance time value used. Then, since $\alpha \neq \tau^*$ we know that the corresponding minimizers of (16) will differ. Consequently, in (17), the inequality $Cx(\tau) \leq L$ will be active and at the minimum $z^*_\alpha(t)$ we will have $Cz^*_\alpha(\alpha) = L$. It then follows that $x^*_\alpha(t) = z^*_\alpha(t), v^*_\alpha(t) = u^*_\alpha(t)$ and $J(z^*_\alpha(t), v^*_\alpha(t)) = J(x^*_\alpha(t), u^*_\alpha(t)) = F(\alpha, g^*(\alpha))$. Then, since $Cx_\beta(\beta) = L$, $\alpha < \beta$ and $Cx(t)$ is monotone, we must have that $Cx^*_\beta(\alpha) < L$ and therefore $x^*_\beta(t), u^*_\beta(t)$ is feasible in (17) with $\tau = \alpha$. Since (17) is strictly convex, and monotonicity gives that $x^*_\beta(t) \neq z^*_\alpha(t)$,$u^*_\beta(t) \neq v^*_\alpha(t)$ we must therefore have that $J(z^*_\alpha(t), v^*_\alpha(t)) < J(x^*_\beta(t), u^*\beta(t)) = F(\beta, g^*(\beta))$ and consequently also $F(\alpha, g(\alpha)) < F(\beta, g(\beta))$ for all $\tau^* < \alpha < \beta < T^h$. Similar results hold for $T^l < \beta < \alpha < \tau^*$, by reversing the inequality in (17). We can therefore conclude that $F(\tau, g^*(\tau))$ is strictly monotone in variations of τ away from τ^*, due to which quasi convexity follows. ■

V. Numerical Example

For illustration we provide an example on how D_F and $F(\tau, \xi)$ can look like for a single vehicle with the characteristics introduced in Section II. We use single input double integrator dynamics with input constraints. The cost is taken as $\Lambda = (v_{ref} - C\dot{x}(t))^2 Q + u^2(t) R$, where v_{ref} is some reference velocity. The dynamics (A, B) are uniformly discretized to (A_d, B_d) and (12b) formulated as a quadratic program over the (discrete) horizon H. As a consequence of discretization, the entrance and exit times can be enforced only on integer multiples of the basic discretization period. Therefore $[T^l, T^h]$ is computed here as

$$T^l = \min_{k:Cx^{max}(k) \geq L} k, \quad T^l = \max_{k:Cx^{min}(k) \leq L} k, \quad (18)$$

where $\{x^{max}(k)\}_{k=1}^H$, $\{x^{min}(k)\}_{k=1}^H$ are retrieved from the sequences obtained by the map

$$x(k+1) = A_d x(k) + B_d \bar{u}, \quad \forall k \in [1, H], \quad x(0) = x_0, \quad (19)$$

using $\bar{u} = u_{max}$ and $\bar{u} = u_{min}$, respectively. Given an entrance time $\tau \in [T^l, T^h]$, the computation of $[g^l(\tau), g^h(\tau)]$ is done according to (18), using sequences $\{x^{max}(k)\}_{k=1}^H$,$\{x^{min}(k)\}_{k=1}^H$

defined by the following two step procedure: 1) The subsequences $\{x^{max}(k)\}_{k=1}^{\tau}$, $\{x^{min}(k)\}_{k=1}^{\tau}$ are computed using linear programming with the objective to maximize and minimize the velocity at $k = \tau$, respectively; 2) $\{x^{max}(k)\}_{k=\tau+1}^{H}$, $\{x^{min}(k)\}_{k=\tau+1}^{H}$ are obtained through

$$x(k+1) = A_d x(k) + B_d \bar{u}, \quad \forall k \in [\tau+1, H] \tag{20}$$

using $\bar{u} = u_{max}$ and $\bar{u} = u_{min}$, respectively. The following problem is then solved repeatedly using CPLEX to cover all feasible values of τ, ξ:

$$\begin{aligned} \underset{x(k),u(k),k=1,\dots,H}{\text{minimize}} \quad & \sum_{k=1}^{H} \Lambda(x(k), u(k)) \\ \text{s.t.} \quad & x(k+1) = A_d x(k) + B_d u(k) \\ & u(k) \in [u_{min}, u_{max}], \\ & Cx(\tau) = L, \; Cx(\xi) = H \end{aligned} \tag{21}$$

The results for a particular instance are given in Fig. 2, obtained by solving 7162 instances of (21) over a grid of feasible τ, ξ. The numerical example supports the theoretical results presented in Section IV and the quasi-convexity of the local cost $F(\tau, \xi)$ is clearly demonstrated. In this example, the vehicle starts at position 0, at its reference velocity 70 *km/h*. The intersection starts after 50 *m* and ends after *70 m*. The cost weights are set as $Q = R = 1$ and the control is bounded to lie within $[-3, 4]$ m/s^2. The horizon is set to $t_f = 5$ s and is discretized into $H = 600$ steps.

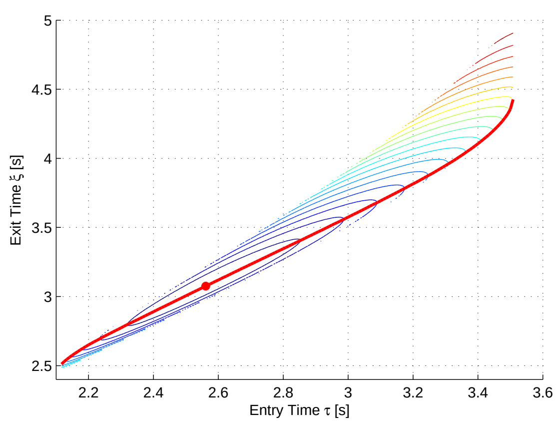


Fig. 2: Visualization of $F(\tau, \xi)$ retrieved by solving (21) for all feasible values of τ, ξ. The curve given by $g^*(\tau)$ is drawn in bold red, and the minimum is marked with a red dot.

VI. Summary and Outlook

In this extended abstract we have considered the problem of coordinating the vehicles crossing a road intersection and formulated it in a constrained optimal control framework. We have shown how to decompose the optimal control problem into a number of smaller subproblems. One *coordination* problem assigns occupancy time slots to each vehicles, i.e., time instants when each vehicle has to enter and clear the intersection, respectively, that minimize a cost function. *Local* problems are then solved to find control input trajectories leading to the assigned occupancy time slots. Further, we have shown important properties of the cost and constraint functions of the local and, therefore, the coordination problems. The obtained results will be used to approximate $F(\tau, \xi)$ and D_F and thus formulate a more tractable coordination problem. Further, we have studied and given results on a particular reduction of D_F, obtained by fixing a relationship between τ and ξ. More precisely, given the entry time τ each vehicle choose their locally optimal exit time $g^*(\tau)$. Comparing the original local problem (12b) with (16) one can then conclude that, under such a reduction, a vehicle can only cooperate *before* it reaches the intersection, and will resort to a “selfish” behaviour once inside. Moreover, preliminary results indicate that $g^*(\tau)$ is strictly monotone and could be conveniently upper bounded with a linear approximation $\tilde{g}(\tau)$, thus allowing slightly longer occupancy times than needed for local optimality. Similarly, the quasi-convexity of $F(\tau, g^*(\tau))$ suggests to approximate it with a convex function $\tilde{F}(\tau)$, e.g., a second order polynomial, centred around its unique minimum.

A convenient formulation of the approximate coordination problem would then be

$$\underset{\tau_i, i \in \mathcal{N}}{\text{minimize}} \quad \sum_{i=1}^{N} \tilde{F}_i(\tau_i) \tag{22a}$$

$$\text{s.t.} \quad \tau_i \in [T_i^l, T_i^h] \tag{22b}$$

$$[\tau_i, \tilde{g}_i(\tau_i)] \cap [\tau_j, \tilde{g}_j(\tau_j)] = \emptyset, \forall i, j, \; i \neq j \tag{22c}$$

where (22a) is the approximated cost, (22b) ensures that the solution is locally feasible and (22c) ensures collision avoidance. The advantage of using such an approach is that the local costs and optimal exit times can be computed a priori. In cooperative scenarios, sharing the approximations $\tilde{F}_i(\tau_i), \tilde{g}(\tau)$ and the interval of feasible entry times $[T^l, T^h]$ with a central computational unit can thus be stated as prerequisites for participation. The optimization in (22) can be stated as a single machine optimal scheduling problem with individual deadlines and release times and job lengths depending on their starting time. To the best of our knowledge, such problems have neither been classified in terms of complexity nor has any efficient algorithm been proposed. For these reasons, it is our intention to address both issues in future work.

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